Theoretische Physik IV: Statistische Mechanik und Thermodynamik

Problem Set No. 3

Due on: Friday, 9.5.08 in the practice groups

Exercise 3.1 *(Stirling approximation)* **(10 points)**

The Stirling approximation is often used in statistical physics (cf. next exercise) because of the factorials, which often occur in combinatorial problems. Derive the Stirling approximation

$$\ln(n!) \approx (n + \frac{1}{2}) \ln(n) - n + \frac{1}{2} \ln(2\pi)$$

in the following way:

(a) First, calculate

$$\int_0^\infty dx e^{-x} x^n$$

Here $n \geq 0$ denotes an integer. (3 points)

(b) Expand the natural logarithm of the above integrand around the position of its maximum and consider the result for large n. Why is it important to expand the logarithm of the integrand and not the integrand itself? (4 points)

(c) Calculate the first correction term. (3 points)

Exercise 3.2 *(Harmonic Oscillators)* **(10 points)**

The energy levels of the harmonic oscillator with frequency ν are given by

$$\epsilon = \frac{1}{2} h\nu, \ldots, (n + \frac{1}{2}) h\nu, \ldots$$

A system of N oscillators has the total energy

$$E = \frac{1}{2} N h\nu + M h\nu$$

(M is an integer).

(a) What is the total number of possible states Ω_M for a given energy E and fixed N? (4 points)

(b) Calculate the entropy in the microcanonical ensemble

$$S = k \ln \Omega_M$$

by means of the Stirling approximation (which reads for large n: $\ln(n!) \approx n \ln n - n$) for $N \gg 1, M \gg 1$ (3 points)

(c) The temperature T is defined as

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

Express the total energy as a function of of temperature and discuss the function $E(T)$. (4 points)

Exercise 3.3 *(Shannon Entropy)* **(10 points)**

(a) Proof that the discrete uniform distribution $p_n = \frac{1}{N}$ $(n = 1, \ldots, N)$ maximizes the Shannon entropy

$$I = -\sum_{n=1}^{N} p_n \log_2 p_n.$$

. (6 points)

(b) Let A and B be two independent one-dimensional systems whose probability density is given by $p(x, y) = p_A(x)p_B(y)$. x and y are the (continuous) microstates of the system A resp. B. How can the total Shannon entropy of both systems I_{ges} be expressed in terms of the Shannon entropy of the system A and system B? Which important property has the entropy therefore? (4 points)